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On the construction of wavefunctions in the six-quark system

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Abstract. A method for calculating the fractional parentage coefficients for many-quark systems in the scheme $U_{CST}(12) \supset U_{CS}(6) \times U_T(2) \supset U_C(3) \times U_S(2) \times U_T(2)$ based on the complementarity of the permutation S_N and unitary $U(n)$ groups, is developed. The scalar factors of the Clebsch–Gordan coefficients for the chain of groups $U(mn) \supset U(m) \times U(n)$ are shown to be independent of the ranks mn, m and n of the groups and to be determined by the Young schemes associated with them. Tables of fractional parentage coefficients for low states of the six-quark (6q) system are presented.

1. Introduction

Much attention has recently been given to a study of the role of quarks in nuclear structure: the problem of dibaryon resonances (Aert 1978, Neudatchin 1977, De Swart 1980), the manifestation of quark structures of nuclei in the cumulative effect (Baldin 1977, Lukyanov and Titov 1979), the derivation of the NN potentials on the basis of quark–quark forces (De Tar 1978, Liberman 1977, Wong 1977), a description of the NN scattering as a scattering of three-quark clusters (Ribeiro 1978, Oka and Yazaki 1980, Toki 1980), the problem of 'hidden' colour in the quark systems (Matveyev and Sorba 1977) and a study of isobar components in the deuteron in terms of the quark model (Smirnov and Tchuvil'sky 1978).

A study of the above problems is reduced to a consideration of the properties of the multi-quark systems. In the simple case of the deuteron and NN scattering it involves a consideration of the 6q system. For this purpose it is first necessary to construct the antisymmetrical wavefunctions of the multi-quark systems in the coloured quark model. This problem is urgent both for the theory of quark bags (De Tar 1978, De Grand 1975) and for the phenomenological non-relativistic quark model (Ribeiro 1978, Liberman 1977).

A conventional method for constructing the wavefunctions of many-particle systems with a given non-relativistic symmetry is the technique of fractional parentage coefficients (FPC). Below we shall consider the problem of calculating FPC for the multi-quark systems and give the tables of their values for the lower states of the system of six non-strange quarks u, d.

According to the Racah lemma (Racah 1949) the FPC is factorised into the orbital, spin, colour and isospin parts. The two latter parts are the same in the bag theory and in the non-relativistic quark models. The orbital and spin parts in the bag theory are like the usual shell FPC in the jj coupling scheme. A disadvantage of the wavefunctions of

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the bag theory is that in these states the centre of mass of the quark system suffers unphysical vibrations.

This disadvantage can be eliminated in the non-relativistic oscillator quark model by analogy with the translationally invariant shell model (TISM) (Neudatchin *et al* 1979). Non-relativistic oscillator quark-model calculations can be carried out using FPC for the TISM calculated by Kurdyumov (1970). Therefore, the present paper is confined mainly to a consideration of the spin–colour–isospin part of the FPC which are the Clebsch–Gordan coefficients (CGC) for the group U(12) in the reduction

$$\begin{aligned} &\mathrm{U}_{CST}(12) \supset \mathrm{U}_{CS}(6) \times \mathrm{U}_T(2), \\ &\mathrm{U}_{CS}(6) \supset \mathrm{U}_C(3) \times \mathrm{U}_S(2), \qquad \mathrm{U}_C(3) \supset \mathrm{O}_C(3). \end{aligned} \tag{1.1}$$

The CGC of the type $\langle q^6 | q^5, q\rangle$, $\langle q^6 | q^3, q^3\rangle$ for the group U(6) in the case of the 6q system were calculated by So and Strottman (1979). We shall be concerned with the CGC of a more general type for the whole chain of groups (1.1) and also the CGC of the type $\langle q^6 | q^4, q^2\rangle$ which are most convenient for the spectroscopic calculation. We use the Weyl method based on the complementarity of the permutation and unitary groups. The complementarity nature of the groups $\mathrm{U}(m)$ and S_N is by now very well known. The first exploitation of results for S_N in calculations for $\mathrm{U}(m)$ are those of Jahn (1950, 1954) and Flowers (1952). These techniques are now textbook material: Vanagas (1971). In our terminology the main idea of the method is the following. The total wavefunction of the system of N quarks $\Psi_n(q^N)$ is comprised of the spatial, spin, colour and isospin parts. If the noted partial wavefunctions are known, the total wavefunction is constructed from them using CGC for the permutation group S_N (see, below, the formula (4.3)). The problem of the fractional parentage expansion of the total wavefunction is therefore reduced to the construction of fractional parentage expansions for its separate parts. The FP expansions for the spin and isospin functions are known (Jahn and van Wieringen 1951). For the colour part of the function it is possible to use the orbital FPC tabulated for the p shell (Jahn and van Wieringen 1951). The FPC for the spatial part of the wavefunction are also easily found. The technique of calculating the Clebsch–Gordan coefficients for the permutation group S_N is described in the monograph by Hammermesh (1964). The corresponding transformational matrices are studied by Kaplan (1969) and tabulated for $N \leqslant 6$ (Kaplan 1962). So, all the components which are necessary and sufficient for constructing the FP expansion of the total wavefunction of the system of N quarks ($N \leqslant 6$) are available in the literature.

We shall present the tables of FPC for the lower states of the 6q systems. Some general properties of CGC for the group $\mathrm{U}(m)$ will be determined.

2. The complementarity of the groups $\mathrm{U}(m)$ and S_N

Let us recall the definition and meaning of the fractional parentage coefficients (FPC) for the part of the wavefunction characterised by the symmetry $\mathrm{U}(m)$ ($m = 2$ for the spin (S) or isospin (T) part, $m = 3$ for the coloured part (C) etc). Suppose we have N particles any of which can be in one of m states $\chi_1, \chi_2, \ldots, \chi_m$. A set of these states forms the basis of a simple irreducible representation (IR) $D^{[1]}$ of the group $\mathrm{U}(m)$ with the Young scheme [1]. The wavefunctions of the system of N particles appear as $\chi_i(1), \chi_j(2), \ldots, \chi_k(N)$, $(i, j, k = 1, 2, \ldots, m)$. The total number of the functions is m^N. In the m-dimensional vector space they form the tensor of rank N. In terms of the group

U(m) they form the basis of a direct product of N irreducible representations $D^{[1]}\times D^{[1]}\times\ldots\times D^{[1]}$, i.e. the question concerns the power $[1]^N$. This representation is reducible and can be expanded into the irreps $D^{[f]}$ of the group U(m)

$$[1]^N=\sum_f \nu_f[f]. \tag{2.1}$$

According to the results of Weyl (1946), this expansion involves the Young schemes $[f]=[f_1f_2\ldots f_m]$ containing N squares and no more than m rows ($N=f_1+f_2+\ldots+f_m$). The multiplicity ν_f of each irrep $D^{[f]}$ in the sum (2.1) is equal to the irrep dimension of the same Young scheme $\{f\}$ but now of the permutation group S_N:

$$\nu_f=\dim\{f\}. \tag{2.2}$$

This results from the fact that the space $\mathscr{L}$ of the representation $[1]^N$ stretched over the components of the tensor of rank N in the m-dimensional space (m^N), can be treated as a space of a certain representation of the direct product of groups $U(m)\times S_N$. If the representation $[1]^N$ is expanded into irreps ($[f]$, $\{f\}$) of these two groups, the result of Weyl (2.1), (2.2) can be rewritten as

$$[1]^N=\sum_f([f],\{f\}). \tag{2.3}$$

This remarkable result shows that in the space $\mathscr{L}$ of the representation $[1]^N$ each irrep $D^{[f]}$ of the group U(m) combines with only one definite irrep of the group S_N with the same Young scheme $\{f\}$. Due to these properties, the groups U(m) and S_N may be called 'complementary' within the space $\mathscr{L}$ (or m^N) following the definition of 'complimentarity' suggested by Moshinsky and Quesne (1970). The noted complementarity accounts for a strong interrelation between different quantities in the representations of the groups U(m) and S_N. These interrelations have been discovered in a variety of recent papers: Kramer and Seligman (1969a, b), Sullivan (1973), Alishauskas (1976), So and Strottman (1979) and Harvey (1981).

Among the earlier results, it is appropriate to mention the construction of fractional parentage coefficients (Jahn 1951, Elliott *et al* 1953) in the nuclear shell model using the matrix elements of the permutation operators $P\in S_N$, which relates, in effect, the CGC of the unitary group to similar quantities of the group S_N. Proceeding from these results, we have studied in more detail the problem of the interrelation between the CGC for the groups U(m) and S_N and have obtained an explicit expression for the CGC of the group U(m) in the reduction of type (1.1) through the CGC for the group S_N. This provides us with a universal method for calculating the CGC for the group U(m) of any rank if the CGC for the inner product of the irreps of the group S_N are known (Hammermesh 1964).

Just as in the papers (Jahn and van Wieringen 1951, Elliott *et al* 1953), we shall use for the representations ($[f]$, $\{f\}$) of the group $U(m)\times S_N$ in the space m^N, the Young–Yamanouchi basis

$$|m^N[f]\alpha,\{f\}(r)\rangle\equiv|m^N[f]\alpha,(r)\rangle. \tag{2.4}$$

Here, (r) is a Yamanouchi symbol specifying a set of the Young permutation schemes $\{f\},\{f^{(N-1)}\},\{f^{(N-2)}\},\ldots,\{f^{(1)}\}=\{1\}$ labelling the irreps of a chain of subgroups

$$S_N\supset S_{N-1}\supset S_{N-2}\supset\ldots\supset S_1. \tag{2.5}$$

α is a complete set of quantum numbers of the group U(m) labelling the vectors of the irrep $D^{[f]}$. However, for this purpose we shall not use the Gel'fand–Tsetlin basis corresponding to the reduction (Gel'fand and Tsetlin 1970)

$$\mathrm{U}(m) \supset \mathrm{U}(m-1) \supset \ldots \supset \mathrm{U}(1)$$

since it is more interesting for physical applications to use the reduction of the type (1.1) into a direct product of subgroups

$$\mathrm{U}_{ab}(m) \supset \mathrm{U}_a(m_a) \times \mathrm{U}_b(m_b), \qquad m = m_a m_b. \tag{2.6}$$

The reduction (2.6) does not provide a complete set of quantum numbers and therefore we use, along with the quantum numbers $([f_a]\alpha_a, [f_b]\alpha_b)$ of the subgroups $\mathrm{U}_a(m_a)$ and $U_b(m_b)$, an additional quantum number, namely the integer index $\omega_{ab} = 1, 2, \ldots$, which will simply enumerate the equivalent representations of the group $\mathrm{U}_a(m_a) \times \mathrm{U}_b(m_b)$ in a given irrep $D^{[f_{ab}]}$ of the group $\mathrm{U}_{ab}(m)$:

$$\alpha_{ab} = \omega_{ab}, ([f_a]\alpha_a, [f_b]\alpha_b). \tag{2.7}$$

It will be noted that in the chain (1.1), a set of indices (ab, a, b) are (CS, C, S), (CST, CS, T) and this sequence can be further continued by combining the orbital space (X) with the CST space since the orbital states can also be specified by the quantum numbers of unitary groups (see § 5).

In order to obtain the fractional parentage expansions in the system of N particles· we should use, along with the Young–Yamanouchi basis (2.4), another basis corresponding to the non-standard reduction

$$\begin{aligned} &\mathrm{S}_N \supset \mathrm{S}_{N'} \times \mathrm{S}_{N''}, \qquad N = N' + N'', \\ &\mathrm{S}_{N'} \supset \mathrm{S}_{N'-1} \supset \ldots \supset \mathrm{S}_1 \\ &\mathrm{S}_{N''} \supset \mathrm{S}_{N''-1} \supset \ldots \supset \mathrm{S}_1. \end{aligned} \tag{2.8}$$

Divide the N particles into two assemblies $1, 2, \ldots, N'$ and $N'+1, N'+2, \ldots, N$ and successively multiply the representations $([f'], \{f'\})$ and $([f''], \{f''\})$ defined by the functions $|m^{N'}[f']\alpha', (r')\rangle$ and $|m^{N''}[f'']\alpha'', (r'')\rangle$ with the aid of the CGC of group U(m)

$$|m^{N'} m^{N''}[f]\alpha, (r'), (r'')\rangle_\gamma = \sum_{\alpha', \alpha''} \left\langle \begin{matrix} [f'] & [f''] \\ \alpha' & \alpha'' \end{matrix} \middle| \begin{matrix} [f] \\ \alpha \end{matrix} \right\rangle_\gamma |m^{N'}[f']\alpha', (r')\rangle |m^{N''}[f'']\alpha'', (r'')\rangle. \tag{2.9}$$

Here, the index γ is the additional quantum number labelling the equivalent representations in the Clebsch–Gordan series for the outer product (Hammermesh 1964) of the Young schemes

$$\{f'\} \times \{f''\} = \sum_f \nu_f \{f\}, \qquad \nu_f \geqslant 1 \tag{2.10}$$

$\gamma = 1, 2, \ldots, \nu_f$. In (2.9) the Clebsch–Gordan coefficients for the group U(m) are assumed to be orthogonalised with respect to the index γ

$$\sum_{\alpha', \alpha''} \left\langle \begin{matrix} [f'] & [f''] \\ \alpha' & \alpha'' \end{matrix} \middle| \begin{matrix} [f] \\ \alpha \end{matrix} \right\rangle_\gamma \left\langle \begin{matrix} [f'] & [f''] \\ \alpha' & \alpha'' \end{matrix} \middle| \begin{matrix} [\bar{f}] \\ \bar{\alpha} \end{matrix} \right\rangle_{\bar{\gamma}} = \delta_{f\bar{f}} \delta_{\alpha\bar{\alpha}} \delta_{\gamma\bar{\gamma}}. \tag{2.11}$$

The basis vectors (2.9) are the linear combinations of vectors (2.4) of the Young–Yamanouchi basis

$$|m^{N'}m^{N''}[f]\alpha, (r'), (r'')\rangle_\gamma = \sum_{\rho''} \langle\{f\}(r=r'\rho'')|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma |m^N[f]\alpha, (r=r'\rho'')\rangle. \tag{2.12}$$

The notation utilised in (2.12) is as follows. The Yamanouchi symbol (r) which consists of N numbers n_i, labelling the rows of the standard Young table $\{f\}$ (i is the particle number)

$$(r) = (n_1 n_2 \ldots n_i \ldots n_{N'} n_{N'+1} \ldots n_N)$$

is divided into two parts. The first part for the particles $1 \leqslant i \leqslant N'$ corresponds to the standard Young table $\{f'\}$ and the Yamanouchi symbol

$$(r') = (n_1 n_2 \ldots n_i \ldots n_{N'})$$

but the second part

$$\rho'' = n_{N'+1} n_{N'+2} \ldots n_N$$

does not correspond to any definite standard Young scheme.

The coefficients in the expansion (2.12) are the transformation matrices (TM) which were introduced by Elliott *et al* (1953) and studied by Kaplan (1962), Kramer (1968) and Kramer and Seligman (1969a, b). For the case $N''=2$ the TM were tabulated by Kaplan (1969). The TM satisfy the following orthogonality relations

$$\sum_{\rho''} \langle\{f\}(r=r'\rho'')|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma \langle\{f\}(r=r'\rho'')|\{f'\}(r'), \{\bar{f}''\}(\bar{r}'')\rangle_{\bar{\gamma}} = \delta_{f''\bar{f}''}\delta_{r''\bar{r}''}\delta_{\gamma\bar{\gamma}}$$

$$\sum_\gamma \sum_{f'',r''} \langle\{f\}(r=r'\rho'')|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma \langle\{f\}(\bar{r}=r'\bar{\rho}'')|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma = \delta_{\rho''\bar{\rho}''} \tag{2.13}$$

that follow from the unitarity of transformation (2.12) with a certain choice of phase factors (when the TMs are real).

It is significant that the TM can be calculated from the equality (2.12) if we operate on the right- and left-hand sides of (2.12) by the operators P of the permutation group S_N using known values of the matrix elements $\langle\{f\}(r)|P|\{f\}(\bar{r})\rangle$ in the Young–Yamanouchi basis (Hammermesh 1964). In the paper (Kaplan 1969), for example, the TM were explicitly expressed in terms of the Young projectors (Jahn 1954)

$$C_{r\bar{r}}^{\{f\}} = \frac{n_f}{N!} \sum_{P\in S_N} \langle\{f\}(r)|P|\{f\}(\bar{r})\rangle P.$$

When the multiplicities are absent ($\nu_f \leqslant 1$ in (2.10))

$$\langle\{f\}(r)|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma = (\langle\{f\}(\bar{r})|C_{\bar{r}''\bar{r}''}^{\{f''\}}|\{f\}(\bar{r})\rangle)^{-1/2}\langle\{f\}(r)|C_{r''\bar{r}''}^{\{f''\}}|\{f\}(\bar{r})\rangle.$$

On the other hand, knowledge of the matrix elements $D_{\alpha\bar{\alpha}}^{[f]}(\mathscr{U}) = \langle[f]\alpha|\mathscr{U}|[f]\bar{\alpha}\rangle$ of the operators $\mathscr{U}$ for the unitary group (D functions) is clearly required (Wigner 1959) to calculate the Clebsch–Gordan coefficients for the unitary group which are needed to construct the same vector in the space $m^{N'} \times m^{N''}$ (2.9). This is a much more complex task as compared with the calculation of $\langle\{f\}(r)|P|\{f\}(\bar{r})\rangle$.

As is already known (Kramer and Seligman 1969a, b, Sullivan 1973, 1978a, b), the matrix elements of permutations in the basis (2.9)

$$\langle m^{N'} m^{N''}[f]\alpha, (r'), (r'')|P|m^{N'} m^{N''}[f]\alpha, (\bar{r}'), (\bar{r}'')\rangle \tag{2.14}$$

are linearly related to the 9*f*-symbols for the unitary group introduced by Kramer (1967).

$$\begin{pmatrix} f_1' & f_2' & f' \\ f_3'' & f_4'' & f'' \\ \bar{f}' & \bar{f}'' & f \end{pmatrix}. \tag{2.15}$$

The 9*f*-symbols (2.15) are an extension of the 9*j*-symbols for the group SU(2) to the case of the unitary group of an arbitrary rank m. The 9*f*-symbols are the convolution of six CGC for the group U(m). The linear relationships of (2.14) to (2.15) establish a direct connection between the CGC for the group U(m) and CGC for the group S_N. This is one of the manifestations of the complementarity of the groups U(m) and S_N in the space m^N. As a result, the 9*f*-symbols (2.15) can be calculated with the formalism of S_N (Kramer 1967, Kramer and Seligman 1969a, b, Sullivan 1973, 1978a, b; Alishauskas 1976). It was noted in the paper (Gurbanovich *et al* 1971) that arbitrary 3*nf*-symbols for the unitary group also depend only on the values of the Young schemes and can be calculated with the formalism of S_N.

In the present paper we propose a concrete method for calculating the CGC for the group U(m) for the reduction (2.6) by means of the formalism of S_N. We shall proceed from the definitions (2.9) and (2.12). Using the orthogonality relations (2.13) the equalities (2.9) and (2.12) are readily rewritten in the form

$$\begin{aligned} &|m^N[f]\alpha, (r = r'\rho'')\rangle \\ &\quad = \sum_{\gamma} \sum_{f'',r''} \sum_{\alpha',\alpha''} \langle\{f\}(r'\rho'')|\{f'\}(r'), \{f''\}(r'')\rangle_\gamma \left\langle \begin{matrix} [f'] & [f''] \\ \alpha' & \alpha'' \end{matrix} \middle| \begin{matrix} [f] \\ \alpha \end{matrix} \right\rangle_\gamma \\ &\quad\quad \times |m^{N'}[f']\alpha', (r')\rangle |m^{N''}[f'']\alpha'', (r'')\rangle. \end{aligned} \tag{2.16}$$

In the next section, using the lemma of factorisation of CGC (Racah 1949), we factorise

$$\left\langle \begin{matrix} [f'] & [f''] \\ \alpha' & \alpha'' \end{matrix} \middle| \begin{matrix} [f] \\ \alpha \end{matrix} \right\rangle_\gamma$$

into terms depending on the Young schemes and in § 4 we express these factors via the TM and CGC for the group S_N. In § 5, the total fractional parentage coefficients in the multi-quark systems are calculated using the CGC for the group $U_{CST}(12)$. It will be noted that it is the fractional parentage expansion of the N-particle wavefunction into the states in the subsystems of N' and N'' particles that is given by the formula (2.16).

3. The scalar factors of the Clebsch–Gordan coefficients for unitary groups

First, consider a simple example, i.e. the CGC for the group $U_C(3)$ in the reduction

$$U_C(3) \supset O_C(3) \supset O_C(2). \tag{3.1}$$

Let C, C_Z be the invariants of the group $O_C(3)$ and $O_C(2)$ (where C is the 'colour moment'). Introducing the index ω_C labelling the equivalent representations C, C_Z in the irrep of group $U_C(3)$ we write down a complete set of the inner quantum numbers $U_C(3)$ as

$$\alpha_C = \omega_C, C, C_z.$$

The basis $|3^{N'}[f'_C]\alpha'_C, (r'_C)\rangle|w^{N''}[f''_C]\alpha''_C, (r''_C)\rangle$ can be reduced in two stages. First, the states with the defined value of C, C_z are obtained using the CGC for the group $O_C(3)$ $(C'C'_zC''C''_z\,|\,CC_z)$. Then the linear combinations of the states obtained are used to obtain the basis of the irrep $D^{[f_C]}$

$$|3^{N'}3^{N''}[f_C]\omega_C CC_z, (r'_C), (r''_C)\rangle_{\gamma_C} = \sum_{\omega'_C, C', \omega''_C, C''} \left\langle \begin{matrix} [f'_C] & [f''_C] \\ \omega'_C, C' & \omega''_C, C'' \end{matrix} \middle\| \begin{matrix} [f_C] \\ \omega_C, C \end{matrix} \right\rangle_{\gamma_C} \sum_{C'_z, C''_z} (C'C'_zC''C''_z\,|\,CC_z) \times|3^{N'}[f'_C]\omega'_C C'C'_z, (r'_C)\rangle|3^{N''}[f''_C]\omega''_C C''C''_z, (r''_C)\rangle. \tag{3.2}$$

The first cofactor in the right-hand side (3.3) is the so-called scalar factor (SF) of CGC for the group $U_C(3)$ in the reduction $U_C(3) \supset O_C(3)$. It depends only on the invariants of the group $U_C(3)$ and $O_C(3)$. If the CGC for the group $U_C(3)$ is known, the SF can be expressed through the CGC by the relation

$$\mathrm{SF}_C \equiv \left\langle \begin{matrix} [f'_C] & [f''_C] \\ \omega'_C, C' & \omega''_C, C'' \end{matrix} \middle\| \begin{matrix} [f_C] \\ \omega_C, C \end{matrix} \right\rangle_{\gamma_C} = \sum_{C'_z, C''_z} (C'C'_zC''C''_z\,|\,CC_z) \left\langle \begin{matrix} [f'_C] & [f''_C] \\ \alpha'_C & \alpha''_C \end{matrix} \middle| \begin{matrix} [f_C] \\ \alpha_C \end{matrix} \right\rangle_{\gamma_C}. \tag{3.3}$$

By analogy with this result we factorise the scalar factors of CGC for the group $U_{CST}(12)$ for all members of the reduction chain (1.1):

(1) SF_C—for the reduction $U_C(3) \supset O_C(3)$, where

$$\alpha_C = \omega_C, C, C_z \tag{3.4}$$

(2) SF_{CS}—for the reduction $U_{CS}(6) \supset U_C(3) \times U_S(2)$, where

$$\alpha_{CS} = \omega_{CS}, ([f_C]\alpha_C, SS_z) \tag{3.5}$$

(3) SF_{CST}—for the reduction $U_{CST}(12) \supset U_{CS}(6) \times U_T(2)$, where

$$\alpha_{CST} = \omega_{CST}, ([f_{CS}]\alpha_{CS}, TT_z). \tag{3.6}$$

It will be shown in the next section that ω_{CS}, ω_{CST} are the multiplicity indices in the inner products of the Young schemes.

As a result, we obtain the following generalisation of (3.2) to the case $U_{CST}(12)$.

$$\left\langle \begin{matrix} [f'_{CST}] & [f''_{CST}] \\ \alpha'_{CST} & \alpha''_{CST} \end{matrix} \middle| \begin{matrix} [f_{CST}] \\ \alpha_{CST} \end{matrix} \right\rangle_{\gamma_{CST}} = \sum_{\gamma_{CS}} \left\langle \begin{matrix} [f'_{CST}] & [f''_{CST}] \\ \omega'_{CST}, ([f'_{CS}], T') & \omega''_{CST}, ([f''_{CS}], T'') \end{matrix} \middle\| \begin{matrix} [f_{CST}] \\ \omega_{CST}, ([f_{CS}], T) \end{matrix} \right\rangle_{\gamma_{CST}\gamma_{CS}} \times (T'T'_zT''T''_z\,|\,TT_z)$$

$$\times\sum_{\gamma_C}\left\langle\begin{matrix}[f'_{CS}] \\ \omega'_{CS},([f'_C],S')\end{matrix}\quad\begin{matrix}[f''_{CS}] \\ \omega''_{CS},([f''_C],S'')\end{matrix}\;\middle\|\;\begin{matrix}[f_{CS}] \\ \omega_{CS},([f_C],S)\end{matrix}\right\rangle_{\gamma_{CS}\gamma_C}$$
$$\times(S'S'_zS''S''_z\,|\,SS_z)$$
$$\times\left\langle\begin{matrix}[f'_C] \\ \omega'_C,C'\end{matrix}\quad\begin{matrix}[f''_C] \\ \omega''_C,C''\end{matrix}\;\middle\|\;\begin{matrix}[f_C] \\ \omega_C,C\end{matrix}\right\rangle_{\gamma_C}(C'C'_zC''C''_z\,|\,CC_z). \tag{3.7}$$

Each of the scalar factors SF_{CST}, SF_{CS}, SF_C can be determined by the relation of type (3.3). The lower line in (3.7) is the CGC for the group $U_C(3)$, the two last lines in (3.7) are the CGC for the group $U_{CS}(6)$, etc. Using the orthogonality relation (2.11) for the CGC of subgroups one can always deduce from (3.7) the relations of type (3.3). For example,

$$SF_{CS}\equiv\left\langle\begin{matrix}[f'_{CS}] \\ \omega'_{CS},([f'_C],S')\end{matrix}\quad\begin{matrix}[f''_{CS}] \\ \omega''_{CS},([f''_C],S'')\end{matrix}\;\middle\|\;\begin{matrix}[f_{CS}] \\ \omega_{CS},([f_C],S)\end{matrix}\right\rangle_{\gamma_{CS}\gamma_C}$$
$$=\sum_{\alpha'_C,\alpha''_C}\sum_{S'_z,S''_z}\left\langle\begin{matrix}[f'_{CS}] \\ \alpha'_{CS}\end{matrix}\quad\begin{matrix}[f''_{CS}] \\ \alpha''_{CS}\end{matrix}\;\middle|\;\begin{matrix}[f_{CS}] \\ \alpha_{CS}\end{matrix}\right\rangle_{\gamma_{CS}}$$
$$\times\left\langle\begin{matrix}[f'_C] \\ \alpha'_C\end{matrix}\quad\begin{matrix}[f''_C] \\ \alpha''_C\end{matrix}\;\middle|\;\begin{matrix}[f_C] \\ \alpha_C\end{matrix}\right\rangle_{\gamma_C}(S'S'_zS''S''_z\,|\,SS_z). \tag{3.8}$$

4. Calculation of the scalar factors

In order to calculate the scalar factor SF_{ab} of the CGC for the unitary group $U_{ab}(m)$ for the chain (2.6) $U_{ab}(m)\supset U_a(m_a)\times U_b(m_b)$ we use the main relation (2.16) from § 2 in the form

$$|(m_am_b)^N[f_{ab}]\alpha_{ab},(r_{ab}=r'_{ab}\rho''_{ab})\rangle$$
$$=\sum_{\gamma_{ab}}\sum_{f''_{ab},r''_{ab}}\sum_{\alpha'_{ab},\alpha''_{ab}}\langle\{f_{ab}\},\rho''_{ab}\,|\{f'_{ab}\},\{f''_{ab}\}(r''_{ab})\rangle_{\gamma_{ab}}$$
$$\times\left\langle\begin{matrix}[f'_{ab}] \\ \alpha'_{ab}\end{matrix}\quad\begin{matrix}[f''_{ab}] \\ \alpha''_{ab}\end{matrix}\;\middle|\;\begin{matrix}[f_{ab}] \\ \alpha_{ab}\end{matrix}\right\rangle_{\gamma_{ab}}$$
$$\times|(m_am_b)^{N'}[f'_{ab}]\alpha'_{ab},(r'_{ab})\rangle|(m_am_b)^{N''}[f''_{ab}]\alpha''_{ab},(r''_{ab})\rangle \tag{4.1}$$

where the CGC for the group $U_{ab}(m)$ will be written as

$$\left\langle\begin{matrix}[f'_{ab}] \\ \alpha'_{ab}\end{matrix}\quad\begin{matrix}[f''_{ab}] \\ \alpha''_{ab}\end{matrix}\;\middle|\;\begin{matrix}[f_{ab}] \\ \alpha_{ab}\end{matrix}\right\rangle_{\gamma_{ab}}$$
$$=\sum_{\gamma_a,\gamma_b}\left\langle\begin{matrix}[f'_{ab}] \\ \omega'_{ab},([f'_a],[f'_b])\end{matrix}\quad\begin{matrix}[f''_{ab}] \\ \omega''_{ab},([f''_a],[f''_b])\end{matrix}\;\middle\|\;\begin{matrix}[f_{ab}] \\ \omega_{ab},([f_a],[f_b])\end{matrix}\right\rangle_{\gamma_{ab}\gamma_a\gamma_b}$$
$$\times\left\langle\begin{matrix}[f'_a] \\ \alpha'_a\end{matrix}\quad\begin{matrix}[f''_a] \\ \alpha''_a\end{matrix}\;\middle|\;\begin{matrix}[f_a] \\ \alpha_a\end{matrix}\right\rangle_{\gamma_a}\left\langle\begin{matrix}[f'_b] \\ \alpha'_b\end{matrix}\quad\begin{matrix}[f''_b] \\ \alpha''_b\end{matrix}\;\middle|\;\begin{matrix}[f_b] \\ \alpha_b\end{matrix}\right\rangle_{\gamma_b}. \tag{4.2}$$

The quantum numbers α_{ab}, α'_{ab}, α''_{ab} are specified in (2.7) and (3.4)–(3.6). The appearance of the indices ω_{ab}, ω'_{ab}, ω''_{ab} will be explained in the following.

The state vector (4.1) can also be determined in the space product $m_a^N \times m_b^N$ by reducing the inner product of the irreps $\{f_a\} \circ \{f_b\}$ using the CGC for the group S_N (Hammermesh 1964)

$$|(m_a m_b)^N [f_{ab}]\alpha_{ab}, (r_{ab})\rangle = \sum_{r_a, r_b} \begin{bmatrix} \{f_a\} & \{f_b\} & \{f_{ab}\} \\ (r_a) & (r_b) & (r_{ab}) \end{bmatrix}_{\omega_{ab}} |m_a^N [f_a]\alpha_a, (r_a)\rangle |m_b^N [f_b]\alpha_b, (r_b)\rangle. \tag{4.3}$$

The corresponding Clebsch–Gordan series for the inner product of the Young schemes

$$\{f_a\} \circ \{f_b\} = \sum_{f_{ab}} \nu_f \{f_{ab}\} \tag{4.4}$$

contains the multiplicities ($\nu_f \geq 1$). In order to distinguish between the equivalent representations $\{f_{ab}\}$ at $\nu_f > 1$ we introduce an index $\omega_{ab} = 1, 2, \ldots, \nu_f$. In this case the CGC for the group S_N can be given in the form orthogonalised with respect to the index ω_{ab} (Hammermesh 1964)

$$\sum_{r_a, r_b} \begin{bmatrix} \{f_a\} & \{f_b\} & \{f_{ab}\} \\ (r_a) & (r_b) & (r_{ab}) \end{bmatrix}_{\omega_{ab}} \begin{bmatrix} \{f_a\} & \{f_b\} & \{\bar{f}_{ab}\} \\ (r_a) & (r_b) & (\bar{r}_{ab}) \end{bmatrix}_{\bar{\omega}_{ab}} = \delta_{f_{ab}\bar{f}_{ab}} \delta_{r_{ab}\bar{r}_{ab}} \delta_{\omega_{ab}\bar{\omega}_{ab}}. \tag{4.5}$$

The sought-for SF_{ab} will be obtained as follows. In the right- and left-hand sides of the main relation (4.1) we go over to an expansion in the elementary basis

$$\Psi_i = |m_a^{N'} [f'_a]\alpha'_a, (r'_a)\rangle |m_a^{N''} [f''_a]\alpha''_a, (r''_a)\rangle |m_b^{N'} [f'_b]\alpha'_b, (r'_b)\rangle |m_b^{N''} [f''_b]\alpha''_b, (r''_b)\rangle. \tag{4.6}$$

For this purpose we insert in the right-hand side of (4.1), the expansions (4.3) in the subspaces $(m_a m_b)^{N'}$ and $(m_a m_b)^{N''}$. The substitutions in the left-hand side of (4.1) are made in two stages. First, the expansion (4.3) in the space $(m_a m_b)^N$ is substituted and then both the vectors in the spaces $(m_a)^N$ and $(m_b)^N$ are expanded in the basis $m_a^{N'} m_a^{N''}$ and $m_b^{N'} m_b^{N''}$ using the relation (2.16). Equating the coefficients at the same functions (4.6) in the right- and left-hand sides of the equality obtained, we arrive at the following set of equations for the CGCs for the group $U_{ab}(m)$

$$\begin{aligned} &\sum_{\rho''_a, \rho''_b} \begin{bmatrix} \{f_a\} & \{f_b\} & \{f_{ab}\} \\ (r_a) & (r_b) & (r_{ab}) \end{bmatrix}_{\omega_{ab}} \sum_{\gamma_a} \langle \{f_a\}, \rho''_a | \{f'_a\}, \{f''_a\}(r''_a) \rangle_{\gamma_a} \left\langle \begin{matrix} [f'_a] & [f''_a] \\ \alpha'_a & \alpha''_a \end{matrix} \middle| \begin{matrix} [f_a] \\ \alpha_a \end{matrix} \right\rangle_{\gamma_a} \\ &\times \sum_{\gamma_b} \langle \{f_b\}, \rho''_b | \{f'_b\}, \{f''_b\}(r''_b) \rangle_{\gamma_b} \left\langle \begin{matrix} [f'_b] & [f''_b] \\ \alpha'_b & \alpha''_b \end{matrix} \middle| \begin{matrix} [f_b] \\ \alpha_b \end{matrix} \right\rangle_{\gamma_b} \\ &= \sum_{\gamma_{ab}} \sum_{f''_{ab}, r''_{ab}} \sum_{\omega'_{ab}, \omega''_{ab}} \langle \{f_{ab}\}, \rho''_{ab} | \{f'_{ab}\}, \{f''_{ab}\}(r''_{ab}) \rangle_{\gamma_{ab}} \\ &\times \left\langle \begin{matrix} [f'_{ab}] & [f''_{ab}] \\ \alpha'_{ab} & \alpha''_{ab} \end{matrix} \middle| \begin{matrix} [f_{ab}] \\ \alpha_{ab} \end{matrix} \right\rangle_{\gamma_{ab}} \begin{bmatrix} \{f'_a\} & \{f'_b\} & \{f'_{ab}\} \\ (r'_a) & (r'_b) & (r'_{ab}) \end{bmatrix}_{\omega'_{ab}} \\ &\times \begin{bmatrix} \{f''_a\} & \{f''_b\} & \{f''_{ab}\} \\ (r''_a) & (r''_b) & (r''_{ab}) \end{bmatrix}_{\omega''_{ab}}. \end{aligned} \tag{4.7}$$

The solution of (4.7) is obtained from the orthogonality properties (2.11), (2.12) and (4.5) and has the form

$$\sum_{\alpha'_a,\alpha''_a}\sum_{\alpha'_b,\alpha''_b}\left\langle\begin{matrix}[f'_{ab}] & [f''_{ab}] \\ \alpha'_{ab} & \alpha''_{ab}\end{matrix}\middle|\begin{matrix}[f_{ab}]\\ \alpha_{ab}\end{matrix}\right\rangle_{\gamma_{ab}}\left\langle\begin{matrix}[f'_a] & [f''_a] \\ \alpha'_a & \alpha''_a\end{matrix}\middle|\begin{matrix}[f_a]\\ \alpha_a\end{matrix}\right\rangle_{\gamma_a}\left\langle\begin{matrix}[f'_b] & [f''_b] \\ \alpha'_b & \alpha''_b\end{matrix}\middle|\begin{matrix}[f_b]\\ \alpha_b\end{matrix}\right\rangle_{\gamma_b}$$

$$=\sum_{r'_a,r'_b}\sum_{r''_a,r''_b}\left[\begin{matrix}\{f'_a\} & \{f'_b\} \\ (r'_a) & (r'_b)\end{matrix}\middle|\begin{matrix}\{f'_{ab}\}\\ (r'_{ab})\end{matrix}\right]_{\omega'_{ab}}\left[\begin{matrix}\{f''_a\} & \{f''_b\} \\ (r''_a) & (r''_b)\end{matrix}\middle|\begin{matrix}\{f''_{ab}\}\\ (r''_{ab})\end{matrix}\right]_{\omega''_{ab}}$$

$$\times\sum_{\rho''_a,\rho''_b,\rho''_{ab}}\left[\begin{matrix}\{f_a\} & \{f_b\} \\ (r'_a\rho''_a) & (r'_b\rho''_b)\end{matrix}\middle|\begin{matrix}\{f_{ab}\}\\ (r'_{ab}\rho''_{ab})\end{matrix}\right]_{\omega_{ab}}\langle\{f_{ab}\},\rho''_{ab}|\{f'_{ab}\},\{f''_{ab}\}(r''_{ab})\rangle_{\gamma_{ab}}$$

$$\times\langle\{f_a\},\rho''_a|\{f'_a\},\{f''_a\}(r''_a)\rangle_{\gamma_a}\langle\{f_b\},\rho''_b|\{f'_b\},\{f''_b\}(r''_b)\rangle_{\gamma_b}. \tag{4.8}$$

The left-hand side of the equality (4.8) is the scalar factor of CGC for the unitary group $U_{ab}(n)$ introduced in (4.2)

$$\mathrm{SF}^{\mathrm{U}}_{ab}=\left\langle\begin{matrix}[f'_{ab}] & [f''_b] \\ \omega'_{ab},([f'_a],[f'_b]) & \omega''_{ab},([f''_a],[f''_b])\end{matrix}\middle\|\begin{matrix}[f_{ab}]\\ \omega_{ab},([f_a],[f_b])\end{matrix}\right\rangle_{\gamma_{ab}\gamma_a\gamma_b} \tag{4.9}$$

The right-hand side of the equality (4.8) contains the analogous quantity (the scalar factor $\mathrm{SF}^{\mathrm{S}}_{ab}$) for the symmetric group S_N. The last two lines of the equality (4.8) may be denoted by:

$$\left[\begin{matrix}\{f_a\} & \{f_b\} \\ (r'_a),(r''_a) & (r'_b),(r''_b)\end{matrix}\middle|\begin{matrix}\{f_{ab}\}\\ (r'_{ab}),(r''_{ab})\end{matrix}\right]^{\gamma_{ab}\gamma_a\gamma_b}_{\omega_{ab}} \tag{4.10}$$

which is actually a CGC for the group S_N for the non-standard reduction (2.8). The right-hand side of the relation (4.8) can now be treated as a definition of the scalar factor of the CGC for the group S_N

$$\mathrm{SF}^{S}_{ab}\equiv\left[\begin{matrix}\{f_a\} & \{f_b\} \\ \gamma_a,(\{f'_a\},\{f''_a\}) & \gamma_b,(\{f'_b\},\{f''_b\})\end{matrix}\middle\|\begin{matrix}\{f_{ab}\}\\ \gamma_{ab},(\{f'_{ab}\},\{f''_{ab}\})\end{matrix}\right]_{\omega_{ab}\omega'_{ab}\omega''_{ab}}$$

$$=\sum_{r'_a,r''_a}\sum_{r'_b,r''_b}\left[\begin{matrix}\{f_a\} & \{f_b\} \\ (r'_a),(r''_a) & (r'_b),(r''_b)\end{matrix}\middle|\begin{matrix}\{f_{ab}\}\\ (r'_{ab}),(r''_{ab})\end{matrix}\right]^{\gamma_{ab}\gamma_a\gamma_b}_{\omega_{ab}}$$

$$\times\left[\begin{matrix}\{f'_a\} & \{f'_b\} \\ (r'_a) & (r'_b)\end{matrix}\middle|\begin{matrix}\{f'_{ab}\}\\ (r'_{ab})\end{matrix}\right]_{\omega'_{ab}}\left[\begin{matrix}\{f''_a\} & \{f''_b\} \\ (r''_a) & (r''_b)\end{matrix}\middle|\begin{matrix}\{f''_{ab}\}\\ (r''_{ab})\end{matrix}\right]_{\omega''_{ab}}. \tag{4.11}$$

Thus, the solution (4.8) to equation (4.7) proves the equality of the scalar factors of the CGC for the unitary and symmetric groups

$$\mathrm{SF}^{\mathrm{U}}_{ab}=\mathrm{SF}^{\mathrm{S}}_{ab}. \tag{4.12}$$

This is a direct consequence of the complementarity of groups $U_{ab}(m)$ and S_N in the space $(m_am_b)^N$.

5. The total fractional parentage coefficient of the *N*q system

The totally antisymmetric state in the system of N quarks can be written in the form

$$\Psi_{\omega\chi f\alpha_{CST}}(q^N)=\sum_{i=1}^{n_f}\frac{\Lambda_i}{(n_f)^{1/2}}\Phi_{\omega\chi f r^{(i)}}(\xi_1,\xi_2,\ldots,\xi_{N-1})|m^N_{CST}[\tilde f]\alpha_{CST}(\tilde r^{(i)})\rangle \tag{5.1}$$

where $\Phi_{\omega_X f r^{(i)}}(\xi_1, \xi_2, \ldots, \xi_{N-1})$ is the orbital part of the wavefunction with the permutation symmetry $\{f\}(r^{(i)})$ in the quark coordinates $x_1, x_2, \ldots, x_N$;

$$\xi_j = x_j - \frac{1}{N-j}\sum_{k=N-j+1}^{N} x_k, \qquad X = \frac{1}{N}\sum_{j=1}^{N} x_j \tag{5.2}$$

are the relative quark coordinates and the centre of mass coordinate; ω_x are additional quantum numbers; n_f is the dimension of the representation $D^{\{f\}}$ of the group S_N; $\Lambda_i/\sqrt{n_f}$ is the Clebsch–Gordan coefficient for the group S_N (Hammermesh 1964) for the totally antisymmetric state in the inner product of the conjugated Young schemes $\{f\}\circ\{\tilde{f}\}\to\{1^N\}$; $\Lambda_i = \pm 1$ is the phase factor of the CGC.

We define the total fractional parentage coefficients (FPC) of the separation of N'' particles $(q^N \to q^{N'} \times q^{N''})$ as the coefficients in the expansion of the function (5.1) as a set of products of totally antisymmetric states in the subsystems $q^{N'}$ and $q^{N''}$ (Neudatchin 1979, Kurdumov 1970)

$$\Psi_{\omega_X f \alpha_{CST}}(q^N) = \sum_{\omega'_X, f', \alpha'_{CST}} \sum_{\omega''_X, f'', \alpha''_{CST}} \sum_{n,l} \Gamma^{nl, \omega_X f \alpha_{CST}}_{\omega'_X f' \alpha'_{CST}, \omega''_X f'' \alpha''_{CST}}$$
$$\times \{\varphi_{nl}(R)\{\Psi_{\omega'_X f' \alpha'_{CST}}(q^{N'})\Psi_{\omega''_X f'' \alpha''_{CST}}(q^{N''})\}_{CST}\}_{\eta L}. \tag{5.3}$$

The braces $\{\quad\}_{CST}$ in (5.3) mean the expansion of functions with the defined values of all three-dimensional momenta in the CST space $\boldsymbol{C} = \boldsymbol{C}' + \boldsymbol{C}''$, $\boldsymbol{S} = \boldsymbol{S}' + \boldsymbol{S}''$, $\boldsymbol{T} = \boldsymbol{T}' + \boldsymbol{T}''$, and the braces $\{\quad\}_{\eta L}$ mean that among the quantum numbers ω_X is the defined value of the total momentum L and of the main orbital quantum number η. $\varphi_{nl}(R)$ are the functions of relative motion of the N'q and N''q clusters,

$$R = \frac{1}{N'}\sum_{i=1}^{N'} x_i - \frac{1}{N''}\sum_{j=N'+1}^{N} x_j. \tag{5.4}$$

All the functions in (5.3) are assumed to be normalised to 1 and hence the normalisation of Γ is

$$\sum_{\omega'_X, f', \alpha'_{CST}} \sum_{\omega''_X, f'', \alpha''_{CST}} \sum_{n,l} |\Gamma^{nl, w_X f \alpha_{CST}}_{w'_X f' \alpha'_{CST}, \omega''_{CST} f'' \alpha''_{CST}}|^2 = 1. \tag{5.5}$$

From the definitions (5.1), (5.3) and (5.5) and the formula (3.7) it is easy to obtain an expression for the total fractional parentage coefficient

$$\Gamma^{nl, \omega_X f \alpha_{CST}}_{\omega'_X f' \alpha'_{CST}, \omega''_X f'' \alpha''_{CST}}$$
$$= (n_{f'} n_{f''}/n_f)^{1/2} (\Phi_{\omega_X f} \| \{\Phi_{\omega'_X f'} \Phi_{\omega''_X f''} \varphi_{nl}\}_{\eta L})$$
$$\times \sum_{\gamma_{CST}, \gamma_{CS}} \left\langle \begin{matrix} [f'_{CST}] & [f''_{CST}] \\ \omega'_{CST}, ([f'_{CS}], T') & \omega''_{CST}, ([f''_{CS}], T'') \end{matrix} \middle\| \begin{matrix} [f_{CST}] \\ \omega_{CST}, ([f_{CS}], T) \end{matrix} \right\rangle_{\gamma_{CST}\gamma_{CS}}$$
$$\times \sum_{\gamma_C} \left\langle \begin{matrix} [f'_{CS}] & [f''_{CS}] \\ \omega'_{CS}, ([f'_C], S') & \omega''_{CS}, ([f''_C], S'') \end{matrix} \middle\| \begin{matrix} [f_{CS}] \\ \omega_{CS}, ([f_C], S) \end{matrix} \right\rangle_{\gamma_{CS}\gamma_C}$$
$$\times \left\langle \begin{matrix} [f'_C] & [f''_C] \\ \omega'_C, C' & \omega''_C, C'' \end{matrix} \middle\| \begin{matrix} [f_C] \\ \omega_C, C \end{matrix} \right\rangle_{\gamma_C} \tag{5.6}$$

The first two cofactors in the right-hand side of (5.6), like the other ones, are the scalar factors of the CGC for some unitary groups. Consider this problem, using the wavefunctions of the oscillatory shell model (SM). The general case is an N-quark configuration $s^{N_s}p^{N_p}$, $N = N_s + N_p$, where s and p are the oscillatory single-particle 0s and 1p states, whose wavefunctions will be denoted as φ_{00} and $\varphi_{1\mu}$, $\mu = 0, \pm 1$. These four functions form the basis of a simple irreducible representation of group $U_X(4)$, and the N-particle orbital states $\varphi_i(1)\varphi_j(2)\ldots\varphi_k(N)$ are the components of the Nth rank tensor in four-dimensional space. Therefore, we can label the orbital shell states with the quantum numbers of group $U_X(4)$.

$$\tilde{\Phi}_{\omega_X f_X r_X}(x_1, x_2, \ldots, x_N) = |s^{N_s}p^{N_p}[f_X]\alpha_X, (r_X)\rangle_{SM} \tag{5.7}$$

and include the group $U_X(4)$ in the reduction chain (1.1) by adding to it the chains

$$\begin{aligned} &U_{XCST}(48) \supset U_X(4) \times U_{CST}(12) \\ &U_X(4) \supset U_p(3) \supset O_X(3) \supset O_X(2). \end{aligned} \tag{5.8}$$

Here $O_X(3)$ is the three-dimensional rotation group and so the quantum numbers α_X can be written in the form analogous to (3.4)

$$\alpha_x = \tilde{\omega}_x, L, L_z. \tag{5.9}$$

So, the orbital factor in (5.6) in the oscillatory shell model is analogous in form to the factor SF_C, for example, at $n=0$, $l=0$,

$$(\Phi_{\omega f}\|\{\Phi_{\omega' f'}\Phi_{\omega'' f''}\varphi_{00}\}_{N_p L}) = \left\langle \begin{matrix} [f'_X] & [f''_X] \\ \tilde{\omega}'_X, L' & \tilde{\omega}''_X, L'' \end{matrix} \middle\| \begin{matrix} [f_X] \\ \tilde{\omega}_X, L \end{matrix} \right\rangle_{\gamma X}$$

and the coefficient $(n_{f'}n_{f''}/n_f)^{1/2}$ in (5.6) can be treated as a scalar factor of the CGC for the group $U_{XCST}(48) \supset U_X(4) \times U_{CST}(12)$. To see this, let us perform a calculation using the formula (4.8) and obtain

$$\left\langle \begin{matrix} [1^{N'}]_{XCST} & [1^{N''}]_{XCST} \\ ([f']_X, [\tilde{f}']_{CST}) & ([f'']_X, [\tilde{f}'']_{CST}) \end{matrix} \middle\| \begin{matrix} [1^N]_{XCST} \\ ([f]_X, [\tilde{f}]_{CST}) \end{matrix} \right\rangle = (n_{f'}n_{f''}/n_f)^{1/2}.$$

In the orbital factor it is necessary to go over from the shell functions (5.7) $\tilde{\Phi}$ to the translationally invariant functions Φ as in (5.1). To this end, one must factorise in (5.7) the function depending on the CM coordinate X (5.2) and to remove the spurious CM excitations by the rules developed in the translationally invariant shell model (TISM) (Neudatchin *et al* 1979) which eventually leads to an additional factor $(m/n)^{1/2} \leqslant 1$ before the orbital factor

$$SF_X = (\Phi_{\omega f}\|\{\Phi_{\omega' f'}\Phi_{\omega'' f''}\varphi_{nl}\}_{\eta L})_{TISM} = (m/n)^{1/2}(\tilde{\Phi}_{\omega f}\|\{\tilde{\Phi}_{\omega' f'}\tilde{\Phi}_{\omega'' f''}\varphi_{nl}\}_{\eta L})_{SM}. \tag{5.10}$$

Consider a specific example, i.e. the configuration $s^4p^2[42]_X L = 0, 2$. As is known (Neudatchin *et al* 1979), in this case there are no spurious CM excitations and so the orbital fractional parentage coefficients of the TISM and SM are the same ($(m/n)^{1/2} = 1$). We use the quantum numbers of the TISM for the states in the two shells (s and p) in the capacity of ω_X. In this case it is sufficient to identify ω_X with the Young scheme $[f_p] = [f_{p1}f_{p2}f_{p3}]$ in the p shell, $N_p = f_{p1} + f_{p2} + f_{p3} = 2$, $N_s = 4$. No-more-than-two-row Young schemes are found in the subsystem N', N'', N'_p, N''_p, N_p, since we assumed that the orbital Young scheme $[f_X] = [42]$ is two-row. Therefore, the orbital scalar factor (5.10) depends in this case only on the two-row Young schemes and must coincide with the analogous factor for the group U(2). It was shown (Obukhovsky *et al* 1979) that if $[f'_p] = [N'_p]$, $[f''_p] = [N''_p]$, $[f_p] = [N_p]$, then the following relation holds true

$$\begin{aligned} SF_X &\equiv \left\langle \begin{matrix} [f'_X] & [f''_X] \\ [N'_p], L' & [N''_p], L'' \end{matrix} \middle\| \begin{matrix} [f_X] \\ [N_p], L \end{matrix} \right\rangle \\ &= (j'j'_z j''j''_z | jj_z) K_L \end{aligned} \tag{5.11}$$

where

$$j'=\tfrac{1}{2}(f_1'-f_2'),\qquad j_z'=\tfrac{1}{2}(N_s'-N_p'),\qquad j''=\tfrac{1}{2}(f_1''-f_2''),\qquad j_z''=\tfrac{1}{2}(N_s''-N_p''),$$
$$j=\tfrac{1}{2}(f_1-f_2),\qquad j_z=\tfrac{1}{2}(N_s-N_p),\qquad N_s'+N_p'=N',\qquad N_s''+N_p''=N'',$$
$$[f_X']=[f_1'f_2'],\qquad [f_X'']=[f_1''f_2''],\qquad [f_X]=[f_1f_2],$$
$$K_L=[(2L'+1)(2L''+1)(2L+1)]^{1/2}\begin{Bmatrix}0 & L' & L'\\ 0 & L'' & L''\\ 0 & L & L\end{Bmatrix}=1.$$

The formula (5.11) holds true not only for the configuration $s^4p^2[42]_XL=0, 2$ for which it was initially derived, but for an arbitrary configuration $s^{N_s}p^{N_p}$ for the two-line Young scheme $[f_1f_2]_X$ as well. In the cases when the Young schemes in the p shell $[f_p']$, $[f_p'']$, $[f_p]$ contain no more than one line, SF_X can be calculated using the standard technique of the two-shell FPC (see for example Neudatchin and Smirnov 1969).

The formalism developed in §§ 4 and 5, has been used to construct the fractional parentage expansions in the six-quark system. Tables 1, 2, 3 give the values of the scalar factors SF_C, SF_{CS} and SF_{CST} for the expansions $q^6\to q^4\times q^2$ and $q^6\to q^3\times q^3$ in two configurations $s^6[6]_XL=0$ and $s^4p^2[42]_XL=0$ with the deuteron-like quantum numbers in the CST space: $[2^3]_CC=0$, $S=1$, $T=0$. In the configuration $s^6[6]_X$ the Pauli principle permits but one state

$$\Psi_1=|s^6[6]_XL=0,[2^3]_CS=1[2^3]_{CS}T=0[1^6]_{CST}\rangle_{\mathrm{TISM}}$$

but in the configuration $s^4p^2[42]_XL=0$ (or 2) there are already five permitted states that differ only by the Young schemes in the CS space. The permitted Young schemes are contained in the Clebsch–Gordan series for the inner product

$$[2^3]_C\circ[42]_S=[42]_{CS}+[321]_{CS}+[2^3]_{CS}+[31^3]_{CS}+[21^4]_{CS}.$$

Table 1. Scalar factors

$$SF_C=\left\langle\begin{matrix}[f_C'] & [f_C''] \\ C' & C''\end{matrix}\,\middle\|\,\begin{matrix}[f_C]\\ C\end{matrix}\right\rangle$$

for the chain $U_C(3)\supset O_C(3)$.
(a): $N'=3$, $N''=3$, $[f_C]=[2^3]$, $C=0$.
(b): $N'=4$, $N''=2$, $[f_C]=[2^3]$, $C=0$.

(a)

$[f_C']\times[f_C'']$	$[1^3]\times[1^3]$	$[21]\times[21]$	
(C', C'')	(0, 0)	(1, 1)	(2, 2)
SF_C	1	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{5}{8}}$

(b)

$[f_C']\times[f_C'']$	$[21^2]\times[1^2]$	$[2^2]\times[2]$	
(C', C'')	(1, 1)	(0, 0)	(2, 2)
SF_C	1	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{5}{6}}$

Table 2. Scalar factors

$$\mathrm{SF}_{CS} = \left\langle \begin{matrix}[f'_{CS}] \\ ([f'_C], S')\end{matrix} \quad \begin{matrix}[f''_{CS}] \\ ([f''_C], S'')\end{matrix} \middle\| \begin{matrix}[f_{CS}] \\ ([f_C], S)\end{matrix} \right\rangle$$

for the chain $U_{CS}(6) \supset U_C(3) \times U_S(2)$.
(a): $N' = 3$, $N'' = 3$, $[f_{CS}] = [2^3]$, $[f_C] = [2^3]$, $S = 1$.
(b): $N' = 3$, $N'' = 3$, $[f_{CS}] = [42]$, $[f_C] = [2^3]$, $S = 1$.
(c): $N' = 4$, $N'' = 2$, $[f_{CS}] = [2^3]$, $[f_C] = [2^3]$, $S = 1$.
(d): $N' = 4$, $N'' = 2$, $[f_{CS}] = [42]$, $[f_C] = [2^3]$, $S = 1$.

(a)

$[f'_{CS}]\times[f''_{CS}]$	$[1^3]\times[1^3]$		$[21]\times[21]$				
$[f'_C]S'$	$[1^3]\frac{3}{2}$	$[21]\frac{1}{2}$	$[1^3]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$
$[f''_C]S''$	$[1^3]\frac{3}{2}$	$[21]\frac{1}{2}$	$[1^3]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$
SF_{CS}	$-\sqrt{\frac{4}{9}}$	$\sqrt{\frac{5}{9}}$	$\sqrt{\frac{5}{36}}$	$\sqrt{\frac{1}{36}}$	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{5}{18}}$

(b)

$[f'_{CS}]\times[f''_{CS}]$	$[3]\times[3]$	$[3]\times[21]$		$[21]\times[3]$		$[21]\times[21]$				
$[f'_C]S'$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[1^3]\frac{1}{2}$
$[f''_C]S''$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$	$[21]\frac{1}{2}$	$[1^3]\frac{1}{2}$
SF_{CS}	1	$\sqrt{\frac{4}{5}}$	$-\sqrt{\frac{1}{5}}$	$\sqrt{\frac{4}{5}}$	$\sqrt{\frac{1}{5}}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{9}{20}}$

(c)

$[f'_{CS}]\times[f''_{CS}]$	$[2^2]\times[2]$			$[21^2]\times[1^2]$			
$[f'_C]S'$	$[21^2]1$	$[2^2]2$	$[2^2]0$	$[21^2]2$	$[21^2]1$	$[21^2]0$	$[2^2]1$
$[f''_C]S''$	$[1^2]0$	$[2]1$	$[2]1$	$[1^2]1$	$[1^2]1$	$[1^2]1$	$[2]0$
SF_{CS}	$\sqrt{\frac{5}{12}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{2}{27}}$	$\sqrt{\frac{5}{9}}$	$-\sqrt{\frac{5}{54}}$	$-\sqrt{\frac{5}{18}}$

(d)

$[f'_{CS}]\times[f''_{CS}]$	$[4]\times[2]$	$[31]\times[2]$		$[31]\times[1^2]$			$[2^2]\times[2]$		
$[f'_C]S'$	$[2^2]0$	$[2^2]1$	$[21^2]1$	$[2^2]1$	$[21^2]1$	$[21^2]0$	$[2^2]0$	$[2^2]2$	$[21^2]1$
$[f''_C]S''$	$[2]1$	$[2]1$	$[1^2]0$	$[2]0$	$[1^2]1$	$[1^2]1$	$[2]1$	$[2]1$	$[1^2]0$
SF_{CS}	1	$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{9}{10}}$

The most interesting is the state with the Young scheme $[42]_{CS}$ where the colourmagnetic forces of quantum chromodynamics lead to the strongest quark attraction (Obukhovsky *et al* 1979). Therefore we have constructed a set of fractional parentage coefficients for the state

$$\Psi_2 = |s^4p^2[42]_X L = 0, [2^3]_C S = 1[42]_{CS} T = 0[2^2 1^2]_{CST}\rangle_{\mathrm{TISM}}.$$

In both the cases (Ψ_1 and Ψ_2) the multiplicites are absent from any products (inner and outer) of the Young schemes and consequently the additional quantum numbers ω and γ are not needed.

Table 3. Scalar factors

$$\mathrm{SF}_{CST} = \left\langle \begin{matrix} [f'_{CST}] \\ ([f'_{CS}], T') \end{matrix} \quad \begin{matrix} [f''_{CST}] \\ ([f''_{CS}], T'') \end{matrix} \;\middle\|\; \begin{matrix} [f_{CST}] \\ ([f_{CS}], T) \end{matrix} \right\rangle$$

for the chain $\mathrm{U}_{CST}(12) \supset \mathrm{U}_{CS}(6) \times \mathrm{U}_T(2)$ and weight coefficients $(n_{f'}n_{f''}/n_f)^{1/2}$.
(a): $N' = 3$, $N'' = 3$, $[f_{CST}] = [1^6]$, $[f_{CS}] = [2^3]$, $T = 0$.
(b): $N' = 3$, $N'' = 3$, $[f_{CST}] = [2^2 1^2]$, $[f_{CS}] = [42]$, $T = 0$.
(c): $N' = 4$, $N'' = 2$, $[f_{CST}] = [1^6]$, $[f_{CS}] = [2^3]$, $T = 0$.
(d): $N' = 4$, $N'' = 2$, $[f_{CST}] = [2^2 1^2]$, $[f_{CS}] = [42]$, $T = 0$.

(a)

$[f'_{CST}] \times [f''_{CST}]$	$[1^2] \times [1^3]$	
$[f'_{CS}]T'$	$[1^3]\frac{3}{2}$	$[21]\frac{1}{2}$
$[f''_{CS}]T''$	$[1^3]\frac{3}{2}$	$[21]\frac{1}{2}$
SF_{CST}	$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{4}{5}}$
$\sqrt{\frac{n_{f'}n_{f''}}{n_f}}$	1	

(b)

$[f'_{CST}] \times [f''_{CST}]$	$[1^3] \times [1^3]$	$[1^3] \times [21]$		$[21] \times [1^3]$		$[21] \times [21]$				
$[f'_{CS}]T'$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[3]\frac{1}{2}$	$[21]\frac{1}{2}$	$[3]\frac{1}{2}$	$[3]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$
$[f''_{CS}]T''$	$[21]\frac{1}{2}$	$[3]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{1}{2}$	$[3]\frac{1}{2}$	$[21]\frac{1}{2}$	$[3]\frac{1}{2}$	$[21]\frac{1}{2}$	$[21]\frac{3}{2}$
SF_{CST}	1	$-\sqrt{\frac{4}{5}}$	$-\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{4}{5}}$	$\sqrt{\frac{1}{5}}$	$\frac{1}{2}$	$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{9}{20}}$
$\left(\frac{n_{f'}n_{f''}}{n_f}\right)^{1/2}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{2}{9}}$		$\sqrt{\frac{2}{9}}$		$\sqrt{\frac{4}{9}}$				

(c)

$[f'_{CST}] \times [f''_{CST}]$	$[1^4] \times [1^2]$	
$[f'_{CS}]T'$	$[21^2]1$	$[2^2]0$
$[f''_{CS}]T''$	$[1^2]1$	$[2]0$
SF_{CST}	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$
$\left(\frac{n_{f'}n_{f''}}{n_f}\right)^{1/2}$	1	

(d)

$[f'_{CST}] \times [f''_{CST}]$	$[1^4] \times [1^2]$	$[21^2] \times [1^2]$		$[21^2] \times [2]$			$[2^2] \times [1^2]$		
$[f'_{CS}]T'$	$[2^2]0$	$[31]0$	$[31]1$	$[31]1$	$[2^2]1$	$[31]0$	$[2^2]0$	$[31]1$	$[4]0$
$[f''_{CS}]T''$	$[2]0$	$[2]0$	$[1^2]1$	$[2]1$	$[2]1$	$[1^2]0$	$[2]0$	$[1^2]1$	$[2]0$
SF_{CST}	1	$-\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{20}}$	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{2}}$
$\left(\frac{n_{f'}n_{f''}}{n_f}\right)^{1/2}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{3}{9}}$		$\sqrt{\frac{3}{9}}$			$\sqrt{\frac{2}{9}}$		

The fractional parentage coefficients for the configuration s^4p^2 have been independently calculated by Harvey (1981) who used another reduction chain, i.e.

$$U_{CST}(12) \supset U_C(3) \times U_{ST}(4)$$

The reduction (1.1) used in the present paper is more convenient for calculations with the forces of quantum chromodynamics symmetric with respect to the group $U_{CS}(6)$ (Jaffe 1977).

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